Bertrand Russell

INTRODUCTION TO MATHEMATICAL PHILOSOPHY

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PREFACE

This book is intended essentially as an "Introduction," and does not aim at giving an exhaustive discussion of the problems with which it deals. It seemed desirable to set forth certain results, hitherto only available to those who have mastered logical symbolism, in a form offering the minimum of difficulty to the beginner. The utmost endeavour has been made to avoid dogmatism on such questions as are still open to serious doubt, and this endeavour has to some extent dominated the choice of topics considered. The beginnings of mathematical logic are less definitely known than its later portions, but are of at least equal philosophical interest. Much of what is set forth in the following chapters is not properly to be called "philosophy," though the matters concerned were included in philosophy so long as no satisfactory science of them existed. The nature of infinity and continuity, for example, belonged in former days to philosophy, but belongs now to mathematics. Mathematical *philosophy*, in the strict sense, cannot, perhaps, be held to include such definite scientific results as have been obtained in this region; the philosophy of mathematics will naturally be expected to deal with questions on the frontier of knowledge, as to which comparative certainty is not yet attained. But speculation on such questions is hardly likely to be fruitful unless the more scientific parts of the principles of mathematics are known. A book dealing with those parts may, therefore, claim to be an *introduction* to mathematical philosophy, though it can hardly claim, except where it steps outside its province, to be actually dealing with a part of philosophy. It does deal, however, with a body of knowledge which, to those who accept it, appears to invalidate much traditional philosophy, and even a good deal of what is current in the present day. In this way, as well as by its bearing on still unsolved problems, mathematical logic is relevant to philosophy. For this reason, as well as on account of the intrinsic importance of the subject, some purpose may be served by a succinct account of the main results of mathematical logic in a form requiring neither a knowledge of mathematics nor an aptitude for mathematical symbolism. Here, however, as elsewhere, the method is more important than the results, from the point of view of further research; and the method cannot well be explained within the framework of such a book as the following. It is to be hoped that some readers may be sufficiently interested to advance to a study of the method by which mathematical logic can be made helpful in investigating the traditional problems of philosophy. But that is a topic with which the following pages have not attempted to deal.

BERTRAND RUSSELL.

EDITOR'S NOTE

THOSE who, relying on the distinction between Mathematical Philosophy and the Philosophy of Mathematics, think that this book is out of place in the present Library, may be referred to what the author himself says on this head in the Preface. It is not necessary to agree with what he there suggests as to the readjustment of the field of philosophy by the transference from it to mathematics of such problems as those of class, continuity, infinity, in order to perceive the bearing of the definitions and discussions that follow on the work of "traditional philosophy." If philosophers cannot consent to relegate the criticism of these categories to any of the special sciences, it is essential, at any rate, that they should know the precise meaning that the science of mathematics, in which these concepts play so large a part, assigns to them. If, on the other hand, there be mathematicians to whom these definitions and discussions seem to be an elaboration and complication of the simple, it may be well to remind them from the side of philosophy that here, as elsewhere, apparent simplicity may conceal a complexity which it is the business of somebody, whether philosopher or mathematician, or, like the author of this volume, both in one, to unravel.

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Introduction to Mathematical Philosophy

CHAPTER I

THE SERIES OF NATURAL NUMBERS

MATHEMATICS is a study which, when we start from its most familiar portions, may be pursued in either of two opposite directions. The more familiar direction is constructive, towards gradually increasing complexity: from integers to fractions, real numbers, complex numbers; from addition and multiplication to differentiation and integration, and on to higher mathematics. The other direction, which is less familiar, proceeds, by analysing, to greater and greater abstractness and logical simplicity; instead of asking what can be defined and deduced from what is assumed to begin with, we ask instead what more general ideas and principles can be found, in terms of which what was our starting-point can be defined or deduced. It is the fact of pursuing this opposite direction that characterises mathematical philosophy as opposed to ordinary mathematics. But it should be understood that the distinction is one, not in the subject matter, but in the state of mind of the investigator. Early Greek geometers, passing from the empirical rules of Egyptian land-surveying to the general propositions by which those rules were found to be justifiable, and thence to Euclid's axioms and postulates, were engaged in mathematical philosophy, according to the above definition; but when once the axioms and postulates had been reached, their deductive employment, as we find it in Euclid, belonged to mathematics in the ordinary sense. The distinction between mathematics and mathematical philosophy is one which depends upon the interest inspiring the research, and upon the stage which the research has reached; not upon the propositions with which the research is concerned.

We may state the same distinction in another way. The most obvious and easy things in mathematics are not those that come logically at the beginning; they are things that, from the point of view of logical deduction, come somewhere in the middle. Just as the easiest bodies to see are those that are neither very near nor very far, neither very small nor very great, so the easiest conceptions to grasp are those that are neither very complex nor very simple (using "simple" in a *logical* sense). And as we need two sorts of instruments, the telescope and the microscope, for the enlargement of our visual powers, so we need two sorts of instruments for the enlargement of our logical powers, one to take us forward to the higher mathematics, the other to take us backward to the logical foundations of the things that we are inclined to take for granted in mathematics. We shall find that by analysing our ordinary mathematical notions we acquire fresh insight, new powers, and the means of reaching whole new mathematical subjects by adopting fresh lines of advance after our backward journey. It is the purpose of this book to explain mathematical philosophy simply and untechnically, without enlarging upon those portions which are so doubtful or difficult that an elementary treatment is scarcely possible. A full treatment will be found in *Principia Mathematica*;[1] the treatment in the present volume is intended merely as an introduction.

To the average educated person of the present day, the obvious starting-point of mathematics would be the series of whole numbers,

1, 2, 3, 4, . . . etc.

[1] Cambridge University Press, vol. i., 1910; vol. ii., 1911; vol. iii., 1913. By Whitehead and Russell.

Probably only a person with some mathematical knowledge would think of beginning with 0 instead of with 1, but we will presume this degree of knowledge; we will take as our starting-point the series:

$$0, 1, 2, 3, \ldots n, n+1, \ldots$$

and it is this series that we shall mean when we speak of the "series of natural numbers."

It is only at a high stage of civilisation that we could take this series as our starting-point. It must have required many ages to discover that a brace of pheasants and a couple of days were both instances of the number 2: the degree of abstraction involved is far from easy. And the discovery that 1 is a number must have been difficult. As for 0, it is a very recent addition; the Greeks and Romans had no such digit. If we had been embarking upon mathematical philosophy in earlier days, we should have had to start with something less abstract than the series of natural numbers, which we should reach as a stage on our backward journey. When the logical foundations of mathematics have grown more familiar, we shall be able to start further back, at what is now a late stage in our analysis. But for the moment the natural numbers seem to represent what is easiest and most familiar in mathematics.

But though familiar, they are not understood. Very few people are prepared with a definition of what is meant by "number," or "0," or "1." It is not very difficult to see that, starting from 0, any other of the natural numbers can be reached by repeated additions of 1, but we shall have to define what we mean by "adding 1," and what we mean by "repeated." These questions are by no means easy. It was believed until recently that some, at least, of these first notions of arithmetic must be accepted as too simple and primitive to be defined. Since all terms that are defined are defined by means of other terms, it is clear that human knowledge must always be content to accept some terms as intelligible without definition, in order